



Fracture Mechanics

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Stress Intensity Factor

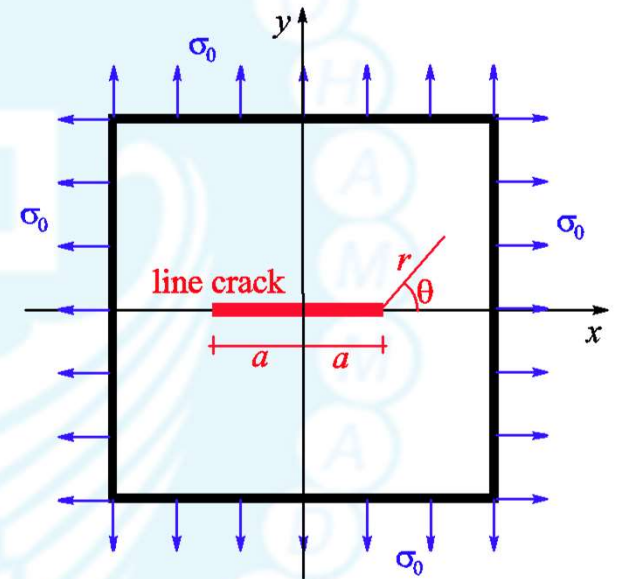
Westergaard analysis of a line crack

The final near crack tip solutions are ($r \ll a$):

$$\sigma_{xx} = \sigma_0 \sqrt{\frac{a}{2r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{yy} = \sigma_0 \sqrt{\frac{a}{2r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{xy} = \sigma_0 \sqrt{\frac{a}{2r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}$$



Irwin (1957) introduced the concept of stress intensity factor (SIF), K , as a measure of the strength of the singularity.

$$K_I = \lim_{\substack{r \rightarrow 0 \\ \theta = 0}} \sigma_{yy} \sqrt{2\pi r}$$

He illustrated that all elastic stress fields around a crack tip are distributed similarly, and K controls the local stress intensity.

Recalling the equations of the elastic stress state around a crack, they can now be represented by:

$$\sigma_{yy} = \sigma_0 \sqrt{\frac{a}{2r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + \dots = \sigma_0 \sqrt{\frac{1}{r}} f(\theta) + \dots$$

For example, the first mode stress intensity factor of the infinite tensile plate with a central crack, can then be simplified to:

$$K_I = \lim_{\substack{r \rightarrow 0 \\ \theta = 0}} \sigma_{yy} \sqrt{2\pi r} = \lim_{\substack{r \rightarrow 0 \\ \theta = 0}} \sqrt{2\pi r} \sigma_0 \sqrt{\frac{a}{2r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) = \sigma_0 \sqrt{\pi a}$$

and:

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

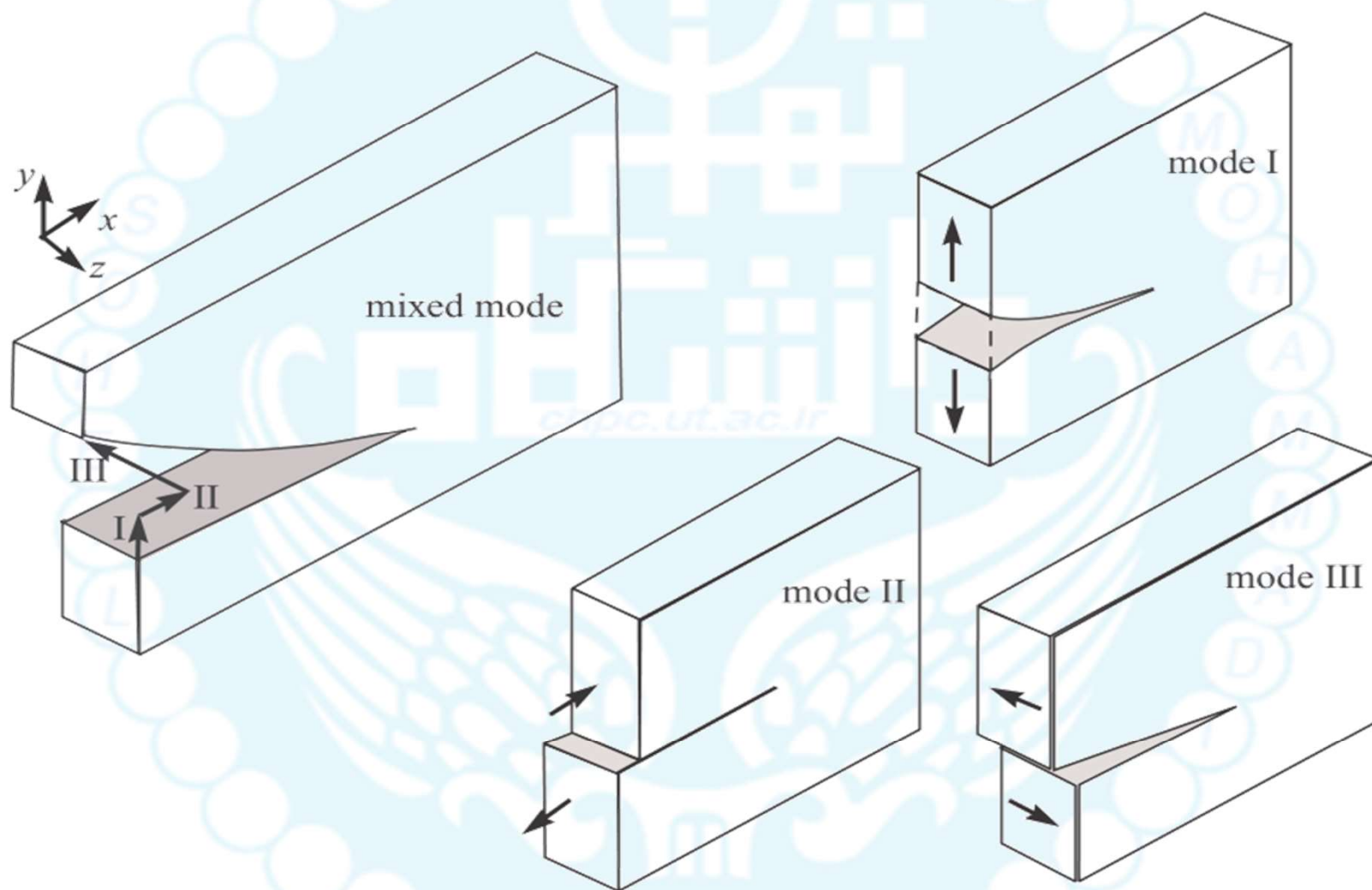
In the stress field:

- K_I represents all effects of geometry, loading boundary conditions, and the crack length.
- The effects of radial and angular variations are independently separated.
- The radial variation is uniquely represented by: $1/\sqrt{r} = r^{-0.5}$
- The angular variation include trigonometric functions in terms of $\theta/2$ and $3\theta/2$.

The stress intensity factor K :

- A measure of intensity of stress concentration
- K is a finite value
- Depends on geometry, loading, boundary and cracking mode
- Independent of material properties for isotropic homogeneous materials
- Can be used to evaluate the state of crack (stability/propagation) if compared to a “Critical Stress Intensity Factor”.
- K has an awkward unit: $MPa\sqrt{m}$

Pure and mixed modes of fracture



or in the more general form of:

$$\sigma_{ij} = r^{-\frac{1}{2}} \left\{ K_{\text{I}} f_{ij}^{\text{I}}(\theta) + K_{\text{II}} f_{ij}^{\text{II}}(\theta) + K_{\text{III}} f_{ij}^{\text{III}}(\theta) \right\} + \text{higher order terms}$$

where

σ_{ij} near crack tip stresses

$K_{\text{I}}, K_{\text{II}}, K_{\text{III}}$ the stress intensity factors associated with three independent modes of movement of crack surfaces

$$K_{\text{I}} = \lim_{\substack{r \rightarrow 0 \\ \theta = 0}} \sigma_{yy} \sqrt{2\pi r}$$

$$K_{\text{II}} = \lim_{\substack{r \rightarrow 0 \\ \theta = 0}} \sigma_{xy} \sqrt{2\pi r}$$

$$K_{\text{III}} = \lim_{\substack{r \rightarrow 0 \\ \theta = 0}} \sigma_{yz} \sqrt{2\pi r}$$

For pure opening mode I, the displacement fields are given by:

$$u_x = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left(\kappa - 1 + 2 \sin^2 \frac{\theta}{2} \right)$$
$$u_y = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left(\kappa + 1 - 2 \cos^2 \frac{\theta}{2} \right)$$
$$\kappa = \begin{cases} \frac{3-\nu}{1+\nu} & \text{plane stress} \\ 3-4\nu & \text{plane strain} \end{cases}$$
$$u_z = \begin{cases} 0 & \text{plane strain} \\ -\frac{\nu z}{E} (\sigma_{xx} + \sigma_{yy}) & \text{plane stress} \end{cases}$$

and the stress field for pure mode I:

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}$$

$$\sigma_{zz} = \begin{cases} \nu(\sigma_{xx} + \sigma_{yy}) & \text{plane strain} \\ 0 & \text{plane stress} \end{cases}$$

$$\sigma_{xz} = \sigma_{yz} = 0$$

and the stress field for pure mode I in polar coordinates:

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin^2 \frac{\theta}{2} \right) = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{5}{4} \cos \frac{\theta}{2} - \frac{1}{4} \cos \frac{3\theta}{2} \right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \cos^3 \frac{\theta}{2} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \right)$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{1}{4} \sin \frac{\theta}{2} + \frac{1}{4} \sin \frac{3\theta}{2} \right)$$

For pure opening mode II, the displacement fields are given by:

$$u_x = \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left(\kappa + 1 + 2 \cos^2 \frac{\theta}{2} \right)$$

$$u_y = -\frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left(\kappa - 1 - 2 \sin^2 \frac{\theta}{2} \right)$$

$$u_z = \begin{cases} 0 & \text{plane strain} \\ -\frac{\nu z}{E} (\sigma_{xx} + \sigma_{yy}) & \text{plane stress} \end{cases}$$

and the stress field for pure mode II:

$$\sigma_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right)$$

$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}$$

$$\sigma_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{zz} = \begin{cases} \nu(\sigma_{xx} + \sigma_{yy}) & \text{plane strain} \\ 0 & \text{plane stress} \end{cases}$$

$$\sigma_{xz} = \sigma_{yz} = 0$$

and the stress field for pure mode II in polar coordinates:

$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left(1 - 3 \sin^2 \frac{\theta}{2} \right) = \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{5}{4} \sin \frac{\theta}{2} + \frac{3}{4} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{\theta\theta} = \frac{-3K_{II}}{2\sqrt{2\pi r}} \sin \theta \cos \frac{\theta}{2} = \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{3}{4} \sin \frac{\theta}{2} - \frac{3}{4} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - 3 \sin^2 \frac{\theta}{2} \right) = \frac{K_{II}}{\sqrt{2\pi r}} \left(\frac{1}{4} \cos \frac{\theta}{2} + \frac{3}{4} \cos \frac{3\theta}{2} \right)$$

The tearing mode III, has only one non-zero displacement and two non-zero stress components:

$$u_x = u_y = 0$$

$$u_z = \frac{K_{\text{III}}}{\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2}$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0$$

$$\sigma_{xz} = -\frac{K_{\text{III}}}{\sqrt{2\pi r}} \sin \frac{\theta}{2}$$

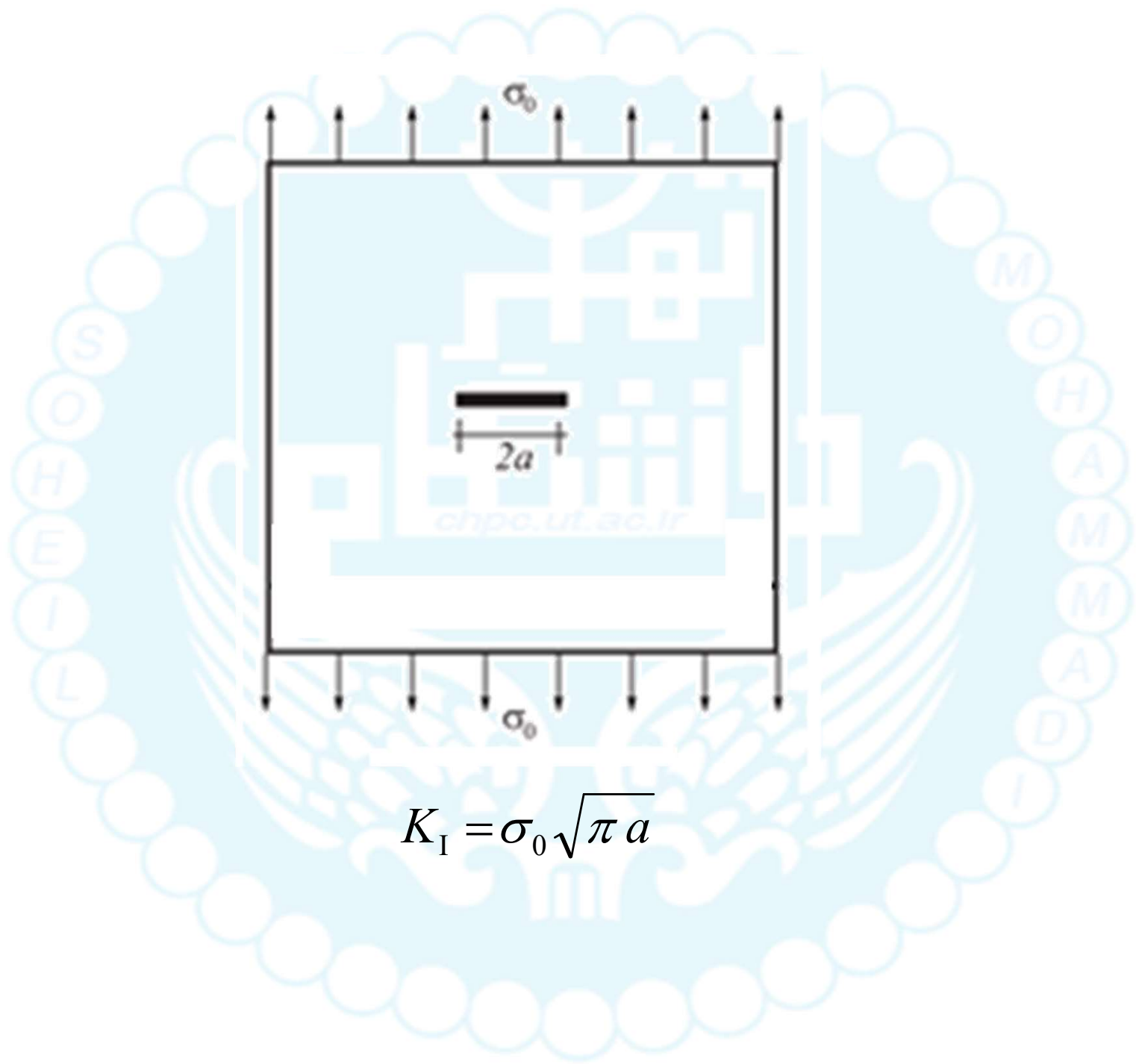
$$\sigma_{yz} = \frac{K_{\text{III}}}{\sqrt{2\pi r}} \cos \frac{\theta}{2}$$

The stress intensity factor can be used to evaluate the state of crack (stability/propagation) if compared to a “Critical Stress Intensity Factor”, also called “Toughness”:

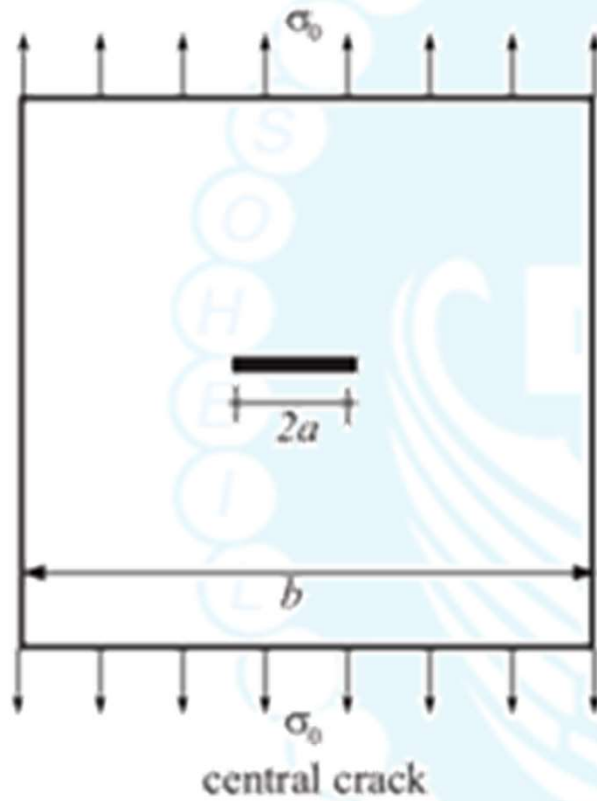
- $K > K_c$ is the simplest crack propagation criterion.
- Toughness, K_c , is a material property and should be measured by laboratory tests.
- While the toughness is assumed to be independent of the size or shape of specimen for conventional engineering problems, it does, in fact, depend to size, shape and even the setup of experiment, and should be cautiously studied in delicate and sensitive applications, where high accuracy is an utmost necessity.



**Examples of
LEFM
Stress Intensity Factors**



$$K_I = \sigma_0 \sqrt{\pi a}$$



Irwin:

$$K_I = \sqrt{\frac{b}{\pi a} \tan \frac{\pi a}{b}} \sigma_0 \sqrt{\pi a}$$

Fedderen:

$$K_I = \sqrt{\sec \frac{\pi a}{b}} \sigma_0 \sqrt{\pi a}$$

Brown:

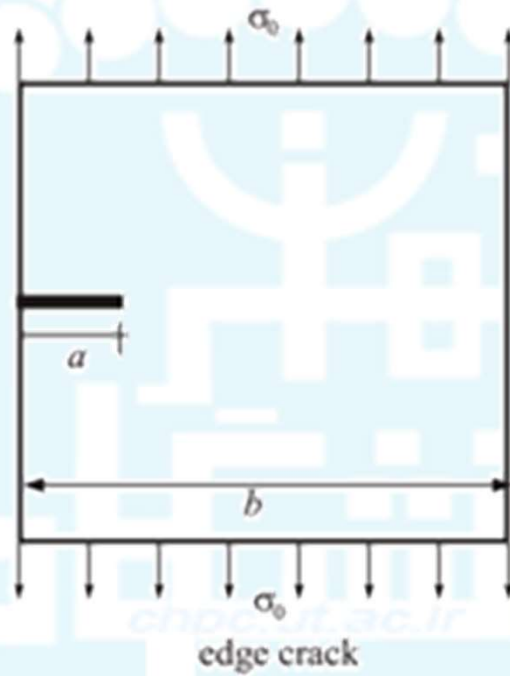
$$K_I = \left[1 + 0.256 \left(\frac{a}{b} \right) - 1.152 \left(\frac{a}{b} \right)^2 + 12.2 \left(\frac{a}{b} \right)^3 \right] \sigma_0 \sqrt{\pi a}$$

Isida:

$$K_I = \left[1 + \sum_{k=1}^7 c_{2k} \left(\frac{2a}{b} \right)^{2k} \right] \sigma_0 \sqrt{\pi a}$$

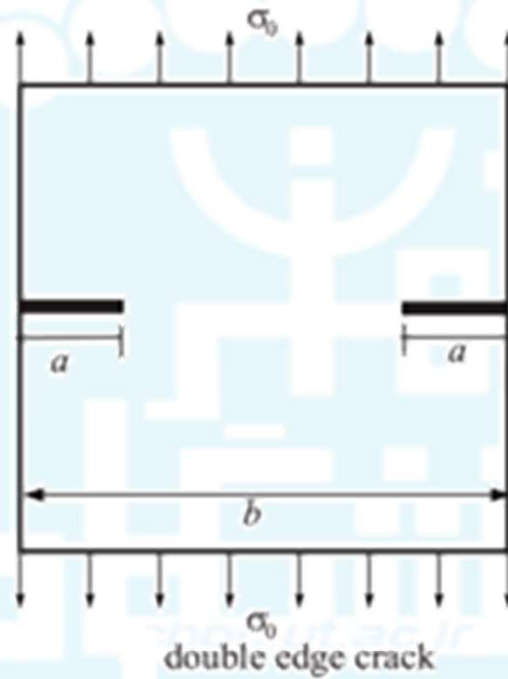
or generally:

$$K_I = \left[1 - 0.1 \left(\frac{a}{b} \right)^2 + 0.96 \left(\frac{a}{b} \right)^4 \right] \sqrt{\sec \frac{\pi a}{b}} \sigma_0 \sqrt{\pi a}$$



$$K_I = \left[1.12 - 0.23 \left(\frac{a}{b} \right) + 10.56 \left(\frac{a}{b} \right)^2 - 21.74 \left(\frac{a}{b} \right)^3 + 30.42 \left(\frac{a}{b} \right)^4 \right] \sigma_0 \sqrt{\pi a}$$

$$a \ll b: \quad \rightarrow \quad K_I = 1.12 \sigma_0 \sqrt{\pi a}$$



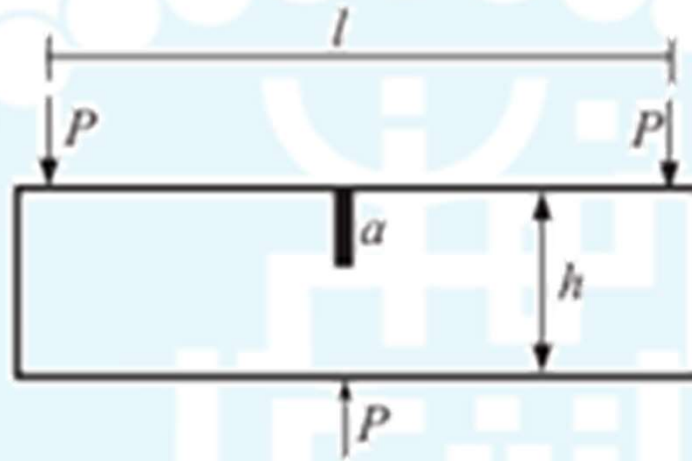
$$K_I = \left[1.12 + 0.43 \left(\frac{a}{b} \right) - 4.79 \left(\frac{a}{b} \right)^2 + 15.46 \left(\frac{a}{b} \right)^3 \right] \sigma_0 \sqrt{\pi a}$$

$$a \ll b: \quad \rightarrow \quad K_I = 1.12 \sigma_0 \sqrt{\pi a}$$



$$K_I = 2\sqrt{3} \frac{Pa}{c^{1.5}}$$

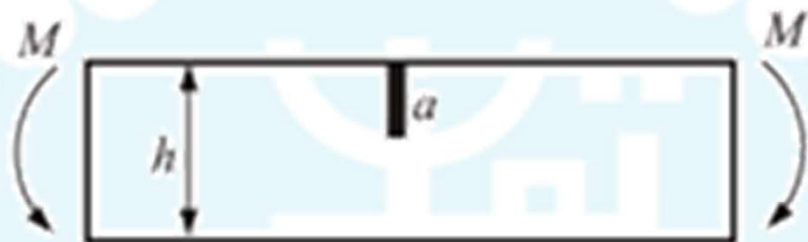
double cantilever beam



edge crack three-point beam

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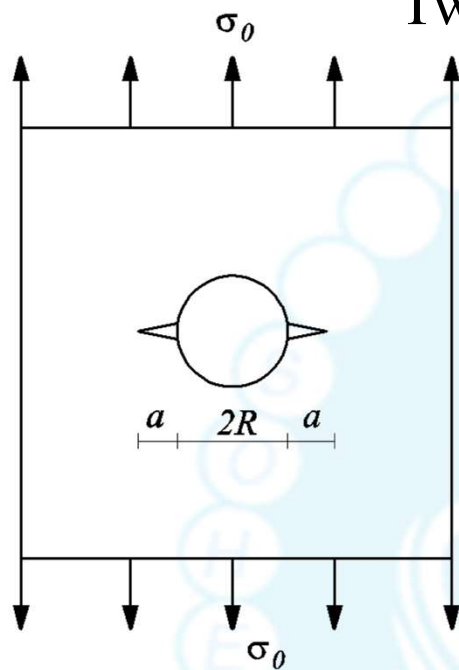
$$K_I = \frac{1.5 \frac{l}{h} \sqrt{\frac{a}{h}}}{\left(1 + 2 \frac{a}{h}\right) \left(1 - \frac{a}{h}\right)^{1.5}} \left[1.99 - \frac{a}{h} \left(1 - \frac{a}{h}\right) \left\{ 2.15 - 3.93 \left(\frac{a}{h}\right) + 2.7 \left(\frac{a}{h}\right)^2 \right\} \right] \frac{P}{th^{0.5}}$$



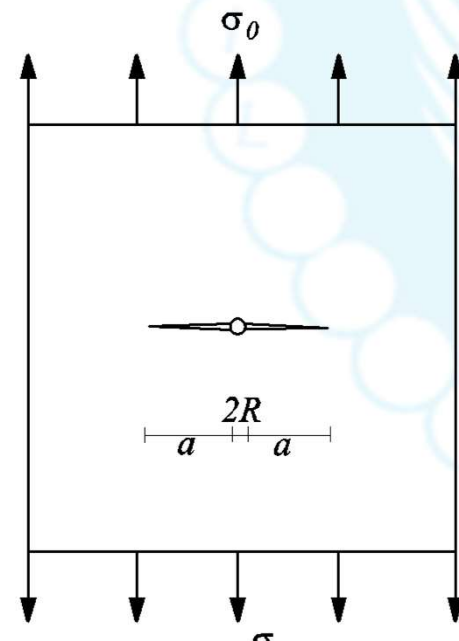
edge crack bending beam

$$K_I = \frac{6M}{th^{1.5}} \frac{\sqrt{2 \tan\left(\frac{\pi a}{2h}\right)}}{\cos\left(\frac{\pi a}{2h}\right)} \left[0.923 + 0.199 \left\{ 1 - \sin\left(\frac{\pi a}{2h}\right) \right\}^4 \right]$$

Two cracks radiating from a circular hole



short crack($\frac{a}{2R} \rightarrow 0$): $K_I = 3.36\sigma_0\sqrt{\pi a}$

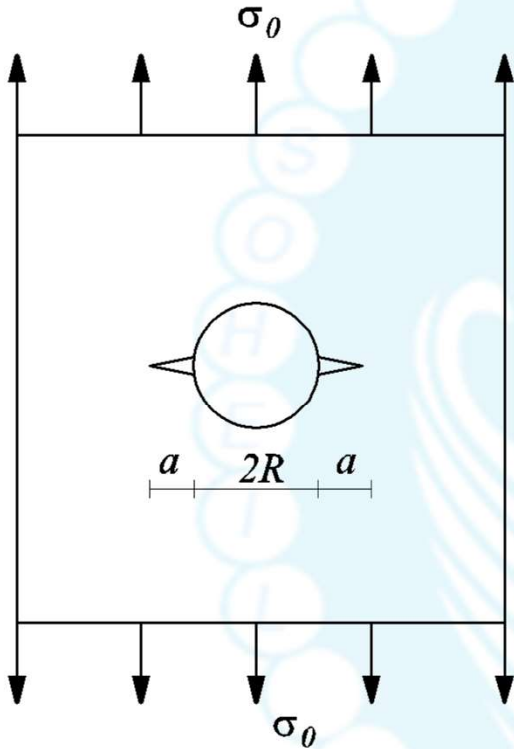


long crack :

$$K_I = \sigma_0\sqrt{\pi a_{eff}}$$

$$2a_{eff} = 2a + 2R$$

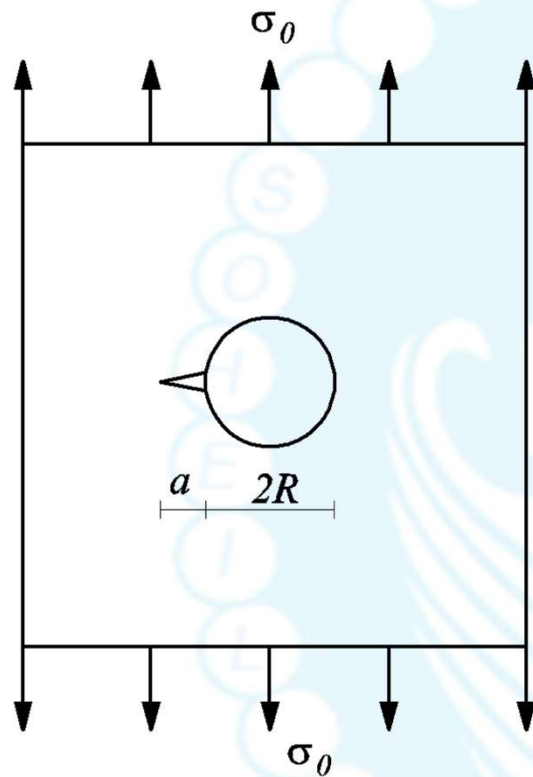
Two cracks radiating from a circular hole



$$K_I = \sigma_0 \sqrt{\pi a_{eff}} = \sigma_0 \sqrt{\pi \frac{2a + 2R}{2}} = \beta \sigma_0 \sqrt{\pi a}$$

Tables for β for other values of $a/2R$

Crack radiating from a circular hole



short crack ($\frac{a}{2R} \rightarrow 0$): $K_I = 3.36\sigma_0\sqrt{\pi a}$

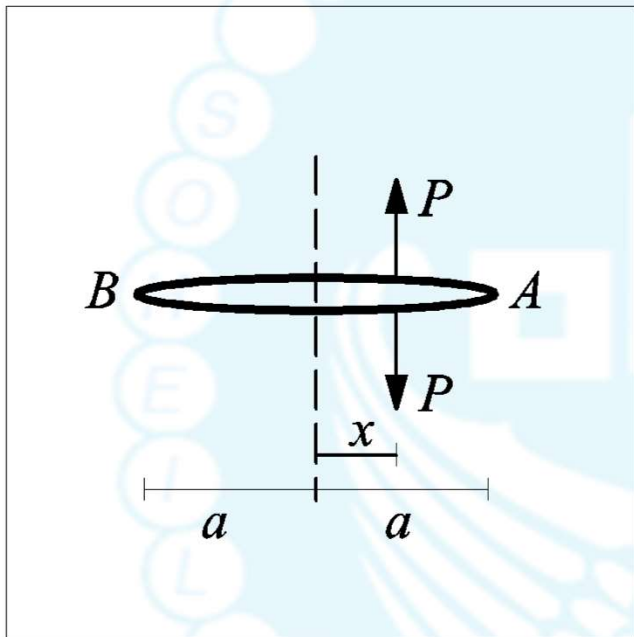
long crack : $K_I = \sigma_0\sqrt{\pi a_{eff}}$

$$2a_{eff} = a + 2R$$

$$K_I = \sigma_0\sqrt{\pi a_{eff}} = \sigma_0\sqrt{\pi \frac{a + 2R}{2}} = \beta\sigma_0\sqrt{\pi a}$$

Tables for β for other values of $a/2R$

Point loads on a center crack



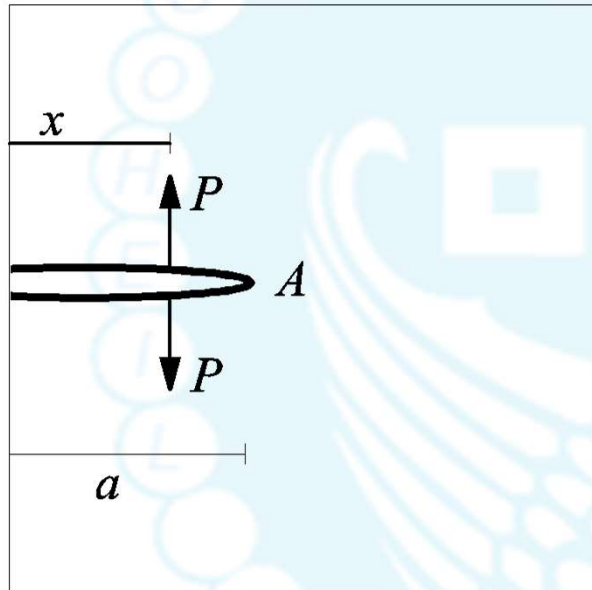
$$K_I^A = \frac{P}{\pi a} \sqrt{\frac{a+x}{a-x}}$$

$$K_I^B = \frac{P}{\pi a} \sqrt{\frac{a-x}{a+x}}$$

Basis for Green functions

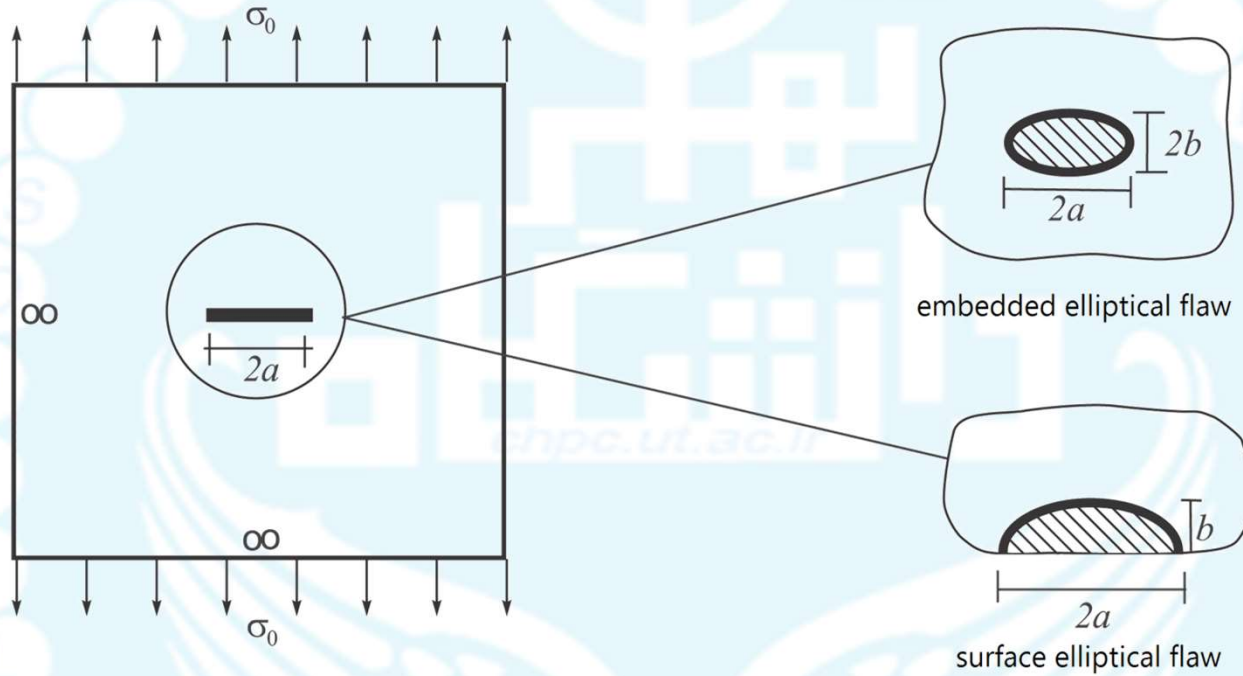
Point loads on an edge crack

$$K_I^A = \frac{2P}{\pi a} \frac{c}{\sqrt{1 + \left(\frac{x}{a}\right)^2}} \left[-0.4\left(\frac{x}{a}\right)^2 + 1.3 \right]$$



x/a	c
< 0.6	1
$0.6 - 0.7$	1.01
$0.7 - 0.8$	1.03
$0.8 - 0.9$	1.07
> 0.9	1.11

Embedded elliptical flaw under normal tension



$$K_I^e(\beta) = \left(1 + 1.464 \left(\frac{b}{a} \right)^{1.65} \right)^{-\frac{1}{2}} \left(\sin^2 \beta + \frac{b^2}{a^2} \cos^2 \beta \right)^{\frac{1}{4}} \sigma_0 \sqrt{\pi b}$$

Embedded elliptical flaw under normal tension

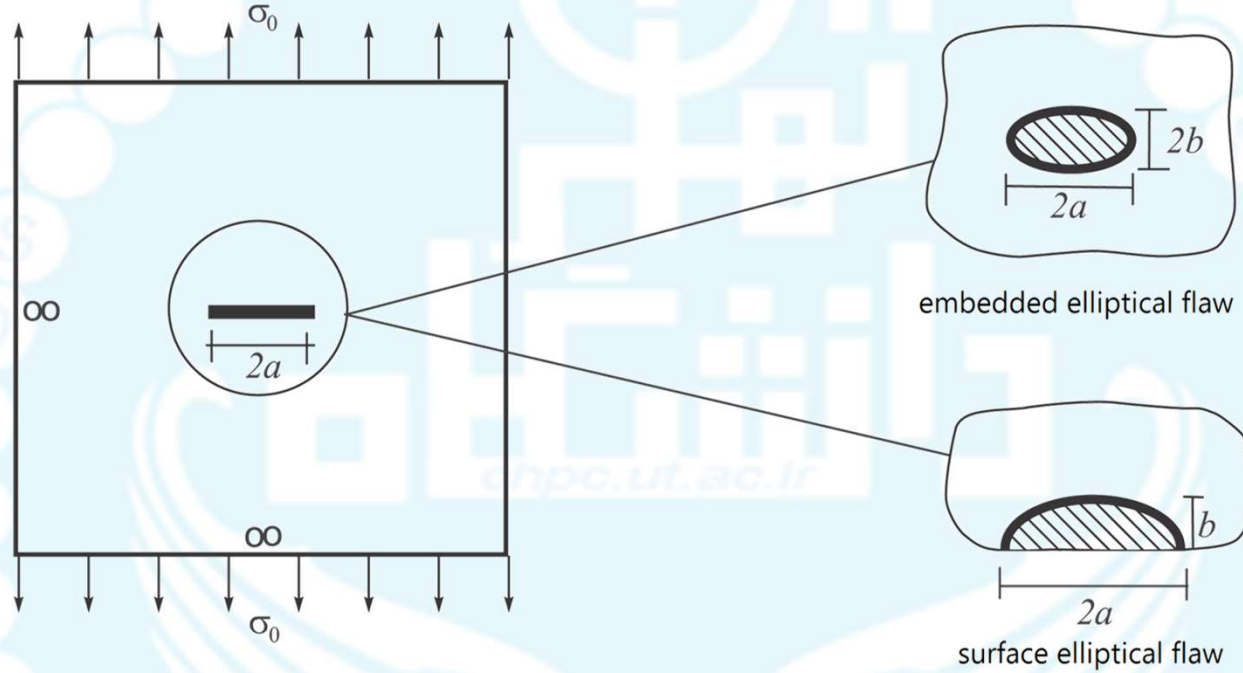
penny – shape crack ($a = b$): $K_I^e(\beta) = \frac{2}{\pi} \sigma_0 \sqrt{\pi a}$

through crack($\frac{a}{b} \rightarrow \infty, \beta = \frac{\pi}{2}$): $K_I^e(\beta) = \sigma_0 \sqrt{\pi a}$

minimum K_I^e : $K_I^e(\beta = 0) \approx \sigma_0 \sqrt{\pi \frac{b^2}{a}}$

maximum K_I^e : $K_I^e(\beta = \frac{\pi}{2}) \approx \sigma_0 \sqrt{\pi b}$

Elliptical surface flaw under normal tension



$$K_I^e(\beta) = \lambda_s \left(1 + 1.464 \left(\frac{b}{a} \right)^{1.65} \right)^{-\frac{1}{2}} \left(\sin^2 \beta + \frac{b^2}{a^2} \cos^2 \beta \right)^{\frac{1}{4}} \sigma_0 \sqrt{\pi b}$$

$$\lambda_s = \left(1.13 - 0.09 \left(\frac{b}{a} \right) \right) \left(1 + 0.1(1 - \sin \beta)^2 \right)$$

Elliptical surface flaw under normal tension

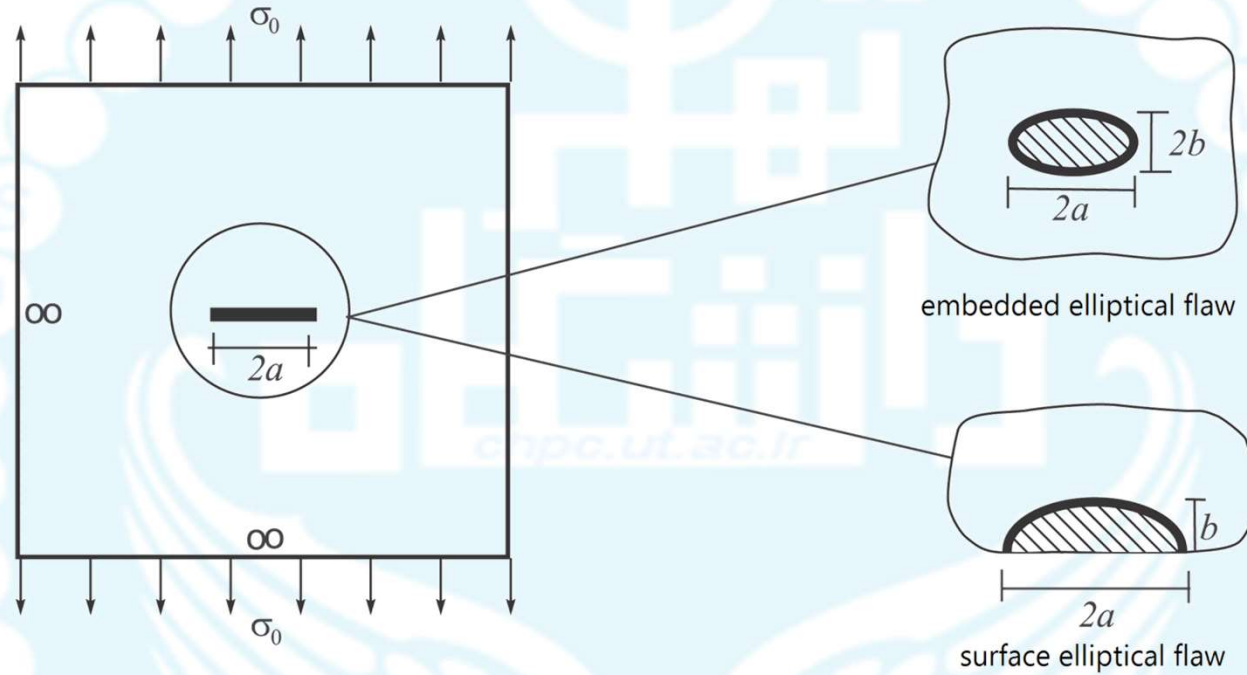
shallow crack ($\frac{b}{t} \ll 1$):

$$K_I^e = 1.13\sigma_0\sqrt{\pi b} \left[1 - 0.08\left(\frac{b}{a}\right) \right] \left[1 + 1.464\left(\frac{b}{a}\right)^{1.65} \right]^{-0.5}$$

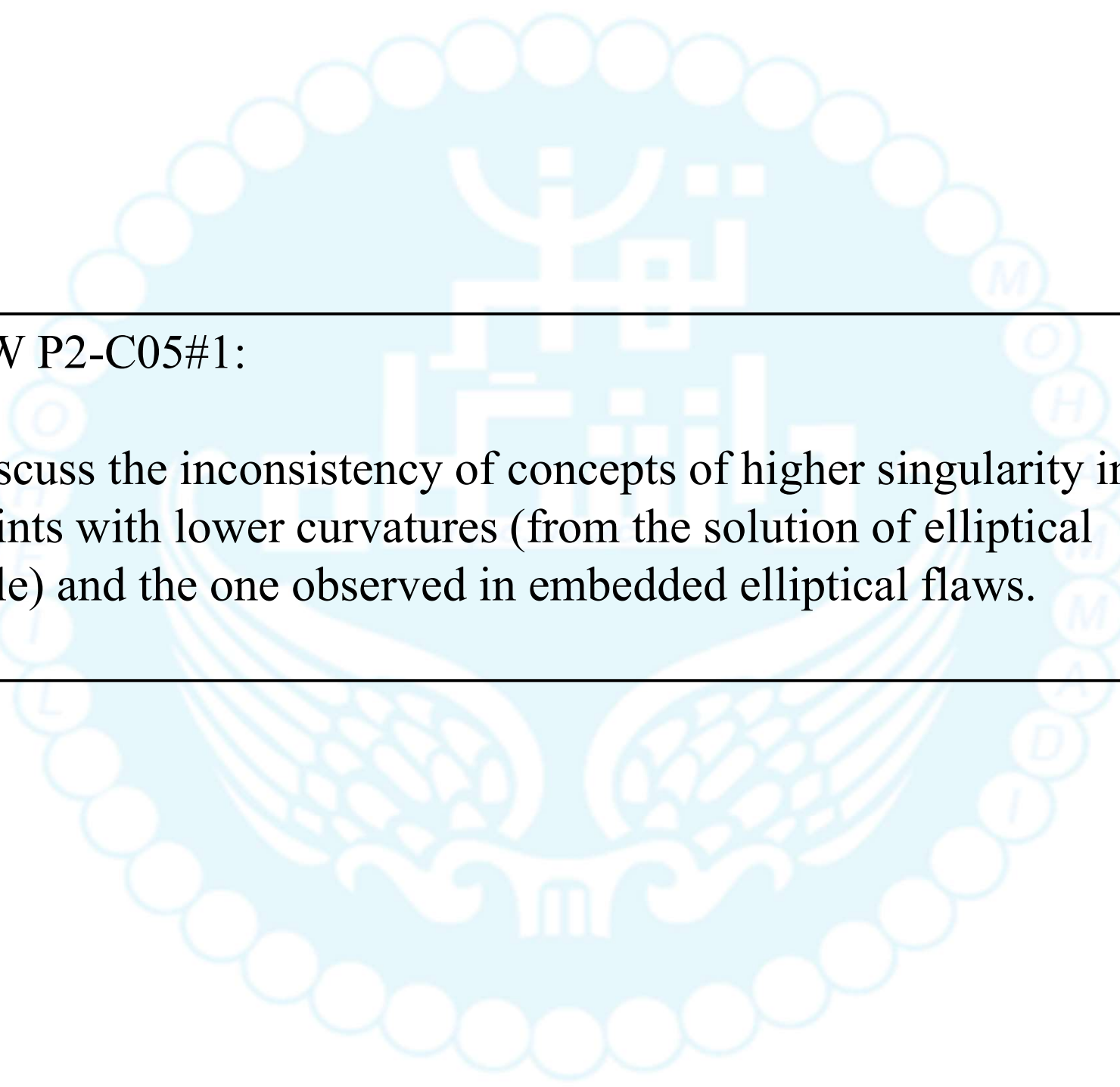
very long crack ($\frac{b}{a} \ll 1$):

$$K_I^e = 1.13\sigma_0\sqrt{\pi b} \left[1 + 3.46\left(\frac{b}{t}\right)^2 + 11.5\left(\frac{b}{t}\right)^4 \right]$$

Embedded/surface elliptical flaw under normal tension



$$\rho(\beta = 0) \ll \rho(\beta = \frac{\pi}{2}) \quad \overset{?}{\rightarrow} \quad K_I^e(\beta = 0) < K_I^e(\beta = \frac{\pi}{2})$$



HW P2-C05#1:

Discuss the inconsistency of concepts of higher singularity in points with lower curvatures (from the solution of elliptical hole) and the one observed in embedded elliptical flaws.

Example:

Assume that a component in the shape of a large sheet is to be fabricated from a specific type of steel, with a decreasing fracture toughness with increase in yield stress (see the Table).

The smallest crack size ($2a$) which can be detected is approximately $.12\text{ in.}$ The specified design stress is $\sigma_y/2$.

To save weight, an increase of tensile strength from 220 ksi to 300 ksi is suggested. Is this reasonable?

Yield Stress Ksi	K_{Ic} ksi $\sqrt{\text{in}}$
210	65
220	60
230	40
240	38
290	35
300	30

$$1 - \sigma_y = 220 :$$

$$\sigma_d = \frac{1}{2} \sigma_y = 110$$

$$K_I = \sigma_d \sqrt{\pi a} = 110 \sqrt{\pi(.06)} = 47.7 < K_c = 60$$

$$2 - \sigma_y = 300 :$$

$$\sigma_d = \frac{1}{2} \sigma_y = 150$$

$$K_I = \sigma_d \sqrt{\pi a} = 150 \sqrt{\pi(.06)} = 65.12 > K_c = 30$$

Therefore, the fracture mechanics concepts does not allow enhancing the steel alloy to $\sigma_y=300$.

HW P2-C05#2:

Obtain the load bearing capacity P of the following beam based on both the strength of material and fracture mechanics:

Figure should be added

